Modelling Intensity of EUR/PLN High Frequency Trade
– a Comparison of ACD-type Models

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Abstract

In the paper a review of alternative parameterizations of the autoregressive conditional duration (ACD) models is presented. We consider several different specifications of the conditional mean of durations as well as the types of conditional distribution. We discuss relative predictive and explanatory performance of a class of competing specifications on the basis of the series of price and trade durations obtained for EUR/PLN exchange rate. To investigate relative performance we present detailed insight into the goodness of fit of estimated models on the basis of probability integral transformation (PIT). The results show that the Log_1 ACD model based on the conditional Burr distribution receives the greatest data support for both price and trade durations under study. The effect of persistence to shocks indicate explosiveness. Although the choice of the Burr distribution results with much more regular histograms of PIT however, the PIT sample is characterized by relatively stronger autocorrelation. In general, models with poor explanatory power exhibit stronger excess of histogram of Z statistics from the uniform distribution but with weaker autocorrelation reported.

1   Introduction

Using the same idea as the one that originated ARCH model for the volatility, (Engle and Russell, 1998) developed the autoregressive conditional duration (ACD) model to describe the evolution of the times between transactions (durations). It was introduced to study transactions data that occur irregularly in time, treating the time between event occurrences as a random process. Most applications of ACD models focus on the analysis of the trading process based on trade and price durations. Initially intensity of trading activity was analysed mainly on equity markets. However there are some exceptions like the paper by Holder et al. (2004) focused on investigation of the price formation process for the futures market. Also Dufour and Engle (2000) analysed transaction data for Treasury Note futures contracts traded at the Chicago Board of Trade. Except of the trading process, some other economic events, such as firm defaults or liquidity traps was subject to research; see Pacurar (2008).

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Foreign exchange market was also subject to analysis through the market microstructure perspective. Fisher and Zurlinden (2004) examine trade intensity with relation to interventions done by central banks on foreign exchange markets. Using daily data on spot transactions of the Federal Reserve, the Bundesbank, and the Swiss National Bank on the dollar market, the authors conclude that traditional variables of a central bank's reaction function for interventions do not improve the ACD specification in their sample.

High Frequency Trading (HFT) as a special case of Algorithmic Trading (AT) become very popular during last decade, particularly of FX market. Hence a detailed insight into empirical properties of ACD-type models is necessary not only from in-sample viewpoint but also regarding forecasts. Consequently in this paper we aim at predictive and explanatory performance of a class of competing ACD specifications on the basis of the series of durations obtained for EUR/PLN trade. To investigate relative performance we report BIC score. We also report measures based on the probability integral transform (PIT) as employed for diagnostics on predictive performance of ACD models.

2   A review of models describing conditional duration

Let denote by $x_i = t_i - t_{i-1}$ the duration between two consecutive events occurred at the time t_i and t_{i-1} respectively. Let F_{i-1} be the information set available until the time t_{i-1}; i.e a series of x_t up to $t = i - 1$. According to Engle and Russel (1998) the duration can be described according to the following formula:

$$x_i = \Psi_i \varepsilon_i, \tag{1}$$

where ε_i is the positive error term with probability density function $f_\varepsilon(\varepsilon_i)$ and $\varepsilon_i \sim IID(1, \sigma_\varepsilon^2)$. In our framework $\Psi_i = \Psi_i(x_i|x_{i-1}, \dots, x_1; \theta) = E(x_i|F_{i-1})$ represents the conditional mean of modelled duration. The original (linear) Autoregressive Conditional Duration model, ACD(p,q) parameterizes the conditional mean duration as follows:

$$\Psi_i = \omega + \sum_{j=1}^{p} \alpha_j x_{i-j} + \sum_{j=1}^{q} \beta_j \Psi_{i-j}. \tag{2}$$

The restrictions: $\omega > 0, \bigwedge_{j=1,2,\dots} \alpha_j \geq 0, \beta_j \geq 0$ ensure positivity of the conditional duration and $\sum_{j=1}^{p} \alpha_j + \sum_{j=1}^{q} \beta_j < 1$ is sufficient for covariance stationarity of the process and existence of the unconditional mean. Here we consider several other specifications of the conditional mean duration Ψ_i

1. Logarithmic ACD$_1$ model – Log$_1$ACD(1,1) – Bauwens and Giot (2006):

$$ln\,\Psi_i = \omega + \alpha_1 \ln \varepsilon_{i-1} + \beta_1 \Psi_{i-1}. \tag{3}$$

2. Logarithmic ACD$_2$ model – Log$_2$ACD(1,1) – (Bauwens & Giot, 2006):

$$ln\,\Psi_i = \omega + \alpha_1\varepsilon_{i-1} + \beta_1\Psi_{i-1}. \tag{4}$$

3. Box-Cox ACD model – BACD(1,1) – (Hautsch, 2001):

$$\Psi_i^{\delta_1} = \omega + \alpha_1\varepsilon_{i-1}^{\delta_2} + \beta_1\Psi_{i-1}^{\delta_1}. \tag{5}$$

4. Augmented Box-Cox ACD model – ABACD(1,1) – (Hautsch, 2012):

$$\Psi_i^{\delta_1} = \omega + \alpha_1\Psi_{i-1}^{\delta_2}[|\varepsilon_{i-1} - \nu| + c_1|\varepsilon_{i-1} - b|]^{\delta_2} + \beta_1\Psi_{i-1}^{\delta_1}. \tag{6}$$

5. Additive and Multiplicative ACD model – AMACD(1,1,1) – Hautsh (2011):

$$\Psi_i = \omega + \alpha_1 x_{i-1} + \nu_1\varepsilon_{i-1} + \beta_1\Psi_{i-1}. \tag{7}$$

Innovations ε_i are positive thus any distribution with positive support could be applied. We consider only distributions that belong to a family of linear-exponential distributions. This useful family of distributions enables to use a quasi-maximum likelihood estimate (QMLE). The estimation procedure is consistent and asymptotically normal irrespective of the actual distribution of a random components. We consider 4 alternative conditional distributions: exponential, Weibull, Generalised Gamma and Burr. The corresponding density functions (assuming that the error terms have unit expectation) and log-likelihood functions are given by:

1. Exponential; see Engle and Russel (1998):

$$f(x_i|\Psi_i; \theta) = \frac{1}{\Psi_i}f_\varepsilon\left(\frac{x_i}{\Psi_i}\right) = \frac{1}{\Psi_i}\exp\left(-\frac{x_i}{\Psi_i}\right), \tag{8}$$

$$\ln L(\theta) = -\sum_{i=1}^{N}\frac{x_i}{\Psi_i}\left(\frac{x_i}{\Psi_i} + \ln\Psi_i\right). \tag{9}$$

2. Weibull; see Engle and Russel (1998):

$$f(x_i|\Psi_i; \theta) = \frac{1}{\Psi_i/\mu}f_\varepsilon\left(\frac{x_i}{\Psi_i/\mu}\right) = \frac{\gamma}{x_i}\left[\frac{x_i\Gamma\left(1+\frac{1}{\gamma}\right)}{\Psi_i}\right]^{\gamma}\exp\left\{-\left[\frac{x_i\Gamma\left(1+\frac{1}{\gamma}\right)}{\Psi_i}\right]^{\gamma}\right\}, \tag{10}$$

$$\ln L(\theta) = \sum_{i=1}^{N}\left\{\ln\left(\frac{\gamma}{x_i}\right) + \gamma\ln\left[\frac{x_i\Gamma\left(1+\frac{1}{\gamma}\right)}{\Psi_i}\right] - \left[\frac{x_i\Gamma\left(1+\frac{1}{\gamma}\right)}{\Psi_i}\right]^{\gamma}\right\}. \tag{11}$$

3. Generalised Gamma; see Lunde (1999):

$$f(x_i|\Psi_i; \theta) = \frac{\gamma}{x_i\Gamma(\nu)}\left[\frac{x_i\Gamma(\nu+1/\gamma)}{\Psi_i\Gamma(\nu)}\right]^{\nu\gamma}\exp\left(-\left[\frac{x_i\Gamma(\nu+1/\gamma)}{\Psi_i\Gamma(\nu)}\right]^{\gamma}\right), \tag{12}$$

$$\ln L(\theta) = \sum_{i=1}^{N}\left\{\ln\left(\frac{\gamma}{x_i\Gamma(\nu)}\right) + \nu\gamma\ln\left[\frac{x_i\Gamma\left(\nu+\frac{1}{\gamma}\right)}{\Psi_i\Gamma(\nu)}\right] - \left[\frac{x_i\Gamma\left(\nu+\frac{1}{\gamma}\right)}{\Psi_i\Gamma(\nu)}\right]^{\gamma}\right\}. \tag{13}$$

4. Burr Gramming and Maurer (2000):

$$f(x_i|\Psi_i; \theta) = \frac{1}{\Psi_i/\mu}f_\varepsilon\left(\frac{x_i}{\Psi_i/\mu}\right) = \frac{\kappa(\Psi_i/\mu)^{-\kappa}(x_i)^{\kappa-1}}{[1+\sigma^2(\Psi_i/\mu)^{-\kappa}(x_i)^{\kappa}]^{\frac{1}{\sigma^2}+1}}, \tag{14}$$

$$\ln L(\theta) = \sum_{i=1}^{N} \left[\ln(\kappa \mu^{\kappa-1}) + (1-\kappa) \ln \Psi_i + (\kappa-1) \ln x_i - \left(1 + \frac{1}{\sigma^2}\right) \ln(1 + \sigma^2 (\mu/$$

$$\Psi_i)^\kappa x_i^\kappa) \right]. \tag{15}$$

3   Empirical analysis

In this section, a set of competing ACD-type specifications are applied to modelling trade intensity of EUR/PLN exchange rate. We analysed the time series of price and trade durations. The price duration is defined as the time until the unit price has changed by at least 8,27E-05 in absolute value, representing the average change of the price of analysed exchange rate. The trade duration is simply the time between two consecutive transactions. The analysed time series cover the time span of 20 workdays: starting from October 6th to 31st of 2014, when transactions recorded only between 9 AM and 5 PM were considered. Transactions that occurred in the same second were aggregated in one with the price referred to latter trade. We assumed that time between two consecutive days equals 1. This resulted with 44 247 price durations and 79 435 trade durations comprising the subject of the analysis. The data were taken from GAIN Capital Group; see http://ratedata.gaincapital.com/.

As described in the literature, durations exhibit intraday patterns; see (Engle and Russell, 1998; Bauwens and Giot, 2006). Following (Engle and Russell, 1998) we transform data by dividing plain intra-daily durations by estimated periodic component. We use Nadaraya-Watson kernel estimator with quadric kernel, considering every week in the sample separately as suggested by Bauwens and Giot (2006). Such method is commonly used by many authors; see Bauwens and Giot (2006), Bień (2006), Chudzicka (2016), Huptas (2009), Białkoska and Pipień (2015).

We present detailed insight into the goodness of fit of estimated models on the basis of probability integral transformation (PIT). In general the purpose of testing procedure proposed by Diebold et al. (1997) was to check if a sequence of one-step-ahead density forecasts of the ACD model is accurate. We utilise this approach in full description of the data fit of competing models. In this paper the PIT is obtained by taking the cumulative distribution function of the residuals given a particular ACD-type specification. Under a proper specification the PIT series are tested to be independently uniformly distributed on the unit interval (0,1):

$$z_i = \int_{-\infty}^{x_i} f_i\left(u | \Psi_{i-1}, \hat{\theta}\right) du, \{z_i\} \sim i.i.d. U(0,1). \tag{16}$$

We perform Kolmogorov-Smirnov test to verify the $i.i.d.\,U(0,1)$ behaviour of $\{z_i\}$. Additionally, we report histograms and correlograms as it is suggested in Diebold et al. (1997).

3.1 Results obtained for price duration

We estimated parameters of 19 models for price durations. According to BIC score presented in Table 1, the conditional Burr distribution is empirically supported as the best choice for all parameterisation of the conditional duration, except of BACD model, the case of the conditional Generalised Gamma distribution receives the best score. Analysing the ranking of competing models with respect to the BIC score the greatest data support seem Log_1ACD with conditional Burr distribution and BACD with Generalised Gamma specification.

In Table 1 we report sum $\alpha_1 + \beta_1$ which represents the effect of persistence to shocks. Among ACD specifications the effect is much stronger in case of exponential and Weibull distribution (the sum $\alpha_1 + \beta_1 = 0.894$ and 0.876 respectively). Much more complicated Burr distribution is able to describe some features of the data in such way the persistence to shocks declines ($\alpha_1 + \beta_1 < 0.7$). Also in case of Log_1ACD the conditional Burr reaches the first place in rank. Surprisingly in case of the best model within analysed subclass this effect indicate explosiveness, as the point estimates of $\alpha_1 + \beta_1$ is greater than 1. Reacher parameterisation of the conditional price duration, proposed as BACD model does not resolve the problem of empirically pervasive effect of persistence to shocks. Close to unity point estimates of underlying sum of parameters is supported in case of three analysed conditional distributions. The best specification, assuming conditional Generalised Gamma distribution is characterised by relatively weaker effect of persistence. For a subclass of ABACD models, for the best model, built on the basis of assumption of the conditional Generalised Gamma distribution the sum is smaller but still close to 1 ($\alpha_1 + \beta_1 = 0.98$). The empirical importance of heavily parameterized conditional distribution in effect of persistence to shocks is also reported in case of AMACD class. Conditionally Burr distribution explains dynamics of the price durations with the use of equation with the weakest effect of persistence as the sum $\alpha_1 + \beta_1$ is smaller than 0.8.

For a detailed insight into differences in explanatory power of the best models (Log_1ACD) we focus on PIT histograms presented in Fig. 1. We calculate Z statistics according to (16) at each data point. Histograms present empirical distribution of Z given ML estimates of parameters. Our results show the role of the conditional Exponential distribution in explaining the distribution of observables can be characterized by huge overestimation of the right tail as the frequencies reach maxima for this region of the distribution function. Also

underestimation of the right tail receives attention, however the nature of this effect is rather different. Histograms exhibit substantial excess from the uniform distribution for the whole region starting from left tail up to the quantile of order 0.25 approximately.

Table 1. Comparison of estimation results and goodness-of-fit.

Model	Distribution	Price durations			Trade durations		
		BIC	$\alpha_1 + \beta_1$	Z	BIC	$\alpha_1 + \beta_1$	Z
ACD	Exponential	65983.75	0.894	0.2049	139080.69	0.925	0.1434
	Weibull	57789.52	0.876	0.0744	135613.98	0.909	0.0343
	Burr	51706.03	0.659	0.0341	119134.51	0.747	0.0941
Log$_1$ ACD	Exponential	67223.94	0.959	0.2241	141042.50	0.988	0.0314
	Weibull	57986.59	0.993	0.0944	136471.90	1.006	0.1096
	Burr	49317.34	1.062	0.0358	111772.70	1.057	0.0396
	Generalized Gamma	53467.00	0.837	0.0728	122304.63	0.710	0.0847
Log$_2$ ACD	Exponential	66381.27	0.986	0.2031	139626.31	1.006	0.1429
	Weibull	58064.70	0.957	0.0784	136083.00	0.994	0.0900
	Burr	52016.72	0.843	0.0382	119658.51	0.874	0.0349
BACD	Exponential	65502.92	1.045	0.2023	138525.39	1.055	0.1469
	Weibull	57407.05	1.060	0.0843	135007.03	1.048	0.0969
	Generalized Gamma	51115.53	0.929	0.0740	123116.54	1.088	0.0937
ABACD	Exponential	65351.88	1.004	0.2055	138487.03	1.062	0.1479
	Weibull	57402.57	1.262	0.0837	134969.30	0.968	0.1021
	Burr	NA	NA	NA	114082.90	1.440	0.0425

| Model	Distribution	Price durations			Trade durations		
		BIC	$\alpha_1 + \beta_1$	Z	BIC	$\alpha_1 + \beta_1$	Z
AMACD	Generalized Gamma	52255.65	0.983	0.0762	NA	NA	NA
	Exponential	65663.44	0.842	0.2014	138784.53	0.901	0.1413
	Weibull	57628.03	0.827	0.0796	135411.95	0.892	0.0917
	Burr	51568.95	0.760	0.0308	118803.62	0.896	0.0339
	Generalized Gamma	NA	NA	NA	127737.70	0.875	0.0803

Conditional Weibull distribution also underestimates both tails of the conditional distribution of price durations. But frequencies corresponding to the left tail indicate much more regular coverage as compared to the case analyzed above. Conditional Burr distribution seems the most regular and it confirms result of model comparison discussed previously. The case of conditional Generalized Gamma distribution generates histograms of Z statistics very similar to the case of the conditional Weibull distribution. The time pattern of dependence of Z can be analyzed on the basis of correlograms presented in Fig. 1. It seems that autocorrelation is relatively higher in case of models with better explanatory power. Also statistically significant correlations were obtained even at 30 and greater, making the case of independence of Z samples improbable in the view of the data. Conditionally Burr distribution, generally supported as the best choice makes histogram more regular however the PIT sample utilized in the procedure is characterized by much stronger autocorrelation.

3.2   Results obtained for trade duration

In case of trade durations we estimated parameters of 20 models. Again, BIC score presented in Table 1 indicates the conditionally Burr Log_1ACD model as the best among all competing specifications. Burr distribution receives the greatest data support for all parameterisation except of BACD model where conditional Generalised Gamma is the most likely.

Just like in case of price durations, when modelling trade durations we report sum $\alpha_1 + \beta_1$ to explain persistence to shocks. Among ACD specifications the conditional Burr distribution receives the strongest data support (the sum $\alpha_1 + \beta_1 < 0.74$). In case of Log1ACD the

conditional Burr reaches the first place in rank again. The sum $\alpha_1 + \beta_1$ is very close to unity for conditionally Exponential model. In case of conditionally Weibull and Burr models this effect indicate explosiveness ($\alpha_1 + \beta_1 > 1$). Equation for conditional trade duration estimated in case of Generalised Gamma distribution can be characterised by the weakest effect of persistence as the point estimated of the sum $\alpha_1 + \beta_1$ slightly crosses 0.8. The class of BACD models is characterised by very strong effect of persistence to shocks. Greater than 1 point estimates of $\alpha_1 + \beta_1$ are obtained in case of all three analysed conditional distributions. For a subclass of ABACD models, for the best model (with Burr distribution), the sum $\alpha_1 + \beta_1$ is much greater than 1 and consequently this case exhibit much stronger effect of persistence to shocks. In case of AMACD class the inference about the sum $\alpha_1 + \beta_1$ is very stable among all four analysed conditional distributions and does not cross the value 0.9. Conditionally Burr distribution explains dynamics of the price durations with the use of equation with the weakest effect of persistence to shocks.
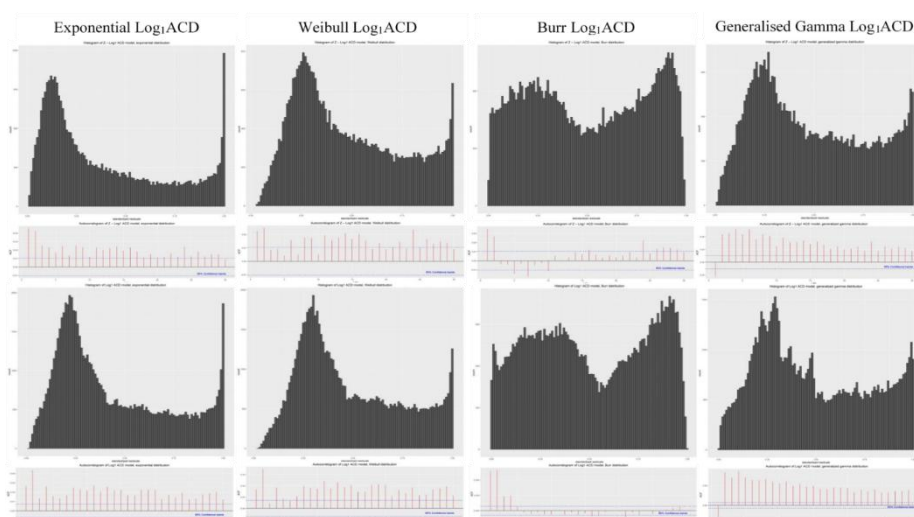
Notes: Top figures relate to price durations, bottom to trade durations.

Fig. 1. Histograms and autocorrelations of Z – the case of Log$_1$ACD models.

Again we calculated PIT histograms and correlograms presented in Fig. 1. Application of different conditional distributions result with qualitatively the same misspecification measured by excess of the histogram of Z statistics from the uniform case. Overestimation of both tails seems the most important feature of the nature of explanatory power of analysed class of models. Left tail receives maxima of frequencies of Z statistics for conditional Exponential distribution. Conditionally Weibull case does not contribute any important information as the histogram obtained for those class is very similar to the class of

Exponential model. Conditional Burr distribution seems the most regular and it confirms its superiority among all discussed competing specifications. The case of conditional Generalised Gamma distribution generates histogram of Z statistics very similar to the case of the conditional Weibull distribution.

Just like in case of series of price durations autocorrelation is higher for models with better fit. Significant correlations were obtained even at 30 and greater, making the case of independence of Z samples rejected. Analyses bring the same results as in previous case.

Conclusions

The paper presents alternative parameterizations of ACD models. We discuss relative predictive and explanatory performance of a class of competing specifications on the basis of the series of price and trade durations obtained for EUR/PLN exchange rate. According to BIC score the conditional Burr distribution receives the greatest data support for both price and trade durations. It is empirically supported as the best choice for most of parameterisation of the conditional duration. In this case the effect of persistence to shocks, measured by the sum $\alpha_1 + \beta_1$ indicate explosiveness. Application of different conditional distributions result in qualitatively the same misspecification measured by excess of the histogram of Z statistics from the uniform case. Only conditional Burr distribution seems the most regular and it confirms its superiority among all competing specifications. The time pattern of dependence of Z analysed on the basis of correlograms show that autocorrelation is higher for models with better fit. Significant correlations were obtained even at 30 and greater, making the case of independence of Z samples rejected. The stochastic nature of trade duration series may be complex in such way the family of the ACD-type models analysed in the paper is too narrow for a proper description and prediction of observed features of price and trade intensity.

Acknowledgement

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