Bayesian inference for deterministic cycle with time-varying amplitude

Łukasz Lenart[1]

Abstract

The main goal of the paper is to obtain posterior distribution for frequency in some generalization of deterministic cycle model proposed by Lenart and Mazur (2016), where the autoregressive model with time-varying almost periodic mean function was investigated with constant amplitude and frequencies. The assumption concerning constant amplitude in such model seems to be too strong to describe the changing nature of the business cycle. Hence, in this paper we assume that the mean function depends on unknown frequencies (related to the length of the cyclical fluctuations) in a similar way as for the almost periodic mean function proposed in Lenart and Mazur (2016), while the assumption concerning constant amplitude was relaxed. More specifically, we assume that the amplitude associated with a given frequency is time-varying and is a linear spline. We obtain the explicit marginal posterior distribution for vector of frequency parameters in the approximate model. We consider real data example concerning monthly production in industry in Poland. The main conclusion is that the posterior for frequency is still likely to be multimodal, but it seems that this multimodality is not as strong as in the deterministic cycle model with constant frequency proposed in Lenart and Mazur (2016).

Keywords: *deterministic cycle, time-varying amplitude, Bayesian inference*

JEL Classification: C22, E32

DOI: 10.14659/SEMF.2018.01.24

1 Introduction

The concept of stochastic cycle is very well known (see Harvey, 2004; Harvey and Jaeger, 1993; Harvey and Trimbur, 2003; Pelagatti, 2016; Trimbur, 2006; Koopman and Shephard, 2015; Azevedo et al., 2006; Harvey et al., 2007 and many others). This concept assumes the stationarity of cyclical fluctuations with zero mean function. The models with a deterministic cycle are not so popular as models with stochastic cycle. Following Harvey (2004) the concept of deterministic cycle is based on almost periodic function at time $t \in \mathbb{Z}$ with one frequency $\lambda \in (0, \pi)$ of the form

$$f(t) = a\sin(\lambda t) + b\cos(\lambda t).$$

It is widely known that above function is not flexible enough to describe the variable in time dynamics of business cycle. Therefore the more flexible concepts were considered. In Lenart and Pipień (2013) the nonparametric inference were considered under assumption that the conditional expectation of observed process contains almost periodic component with

[1] Cracow University of Economics, Department of Mathematics, ul. Rakowicka 27, 31-510 Kraków, Poland, e-mail: lukasz.lenart@uek.krakow.pl.

more than one frequency. In Lenart et al. (2016) the parametric and nonparametric inference were considered under assumption that the mean function of cyclical process is almost periodic function with few frequencies. Finally in Lenart and Pipień (2015, 2017a, 2017b) authors consider the nonaprametric test based on subsampling approach to test the deterministic cycles. In Mazur (2016, 2017a, 2017b) the parametric model containing deterministic cycle was considered. In all above approaches the amplitude of considered deterministic cycle is assumed to be constant in time. This assumption seems to be strong taking into consideration the variable nature of the business cycle. Note that in recent few years the deviation cycle in Poland has flattened.

Therefore, we investigate in this paper the time varying amplitude by considering the following time-varying function:

$$g(\lambda,t) = a(t)\sin(\lambda t) + b(t)\cos(\lambda t) \tag{1}$$

of integer $t \in \{1,2,\dots,n\}$, where $a(t)$ and $b(t)$ are a linear splines with $r+1$ knots $\{(t_i,a_i), i = 0,1,\dots,r\}$ for $a(t)$ and $\{(t_i,b_i), i = 0,1,2,\dots,r\}$ for $b(t)$. We assume $t_0 = 1$ and $t_r = n$. Hence

$$a(t) = \sum_{i=1}^{r} \mathrm{I}_{\{t_{i-1}\le t<t_i\}}\left[a_{i-1}\frac{(t_i-t)}{t_i-t_{i-1}} + a_i\frac{(t-t_{i-1})}{t_i-t_{i-1}}\right], \qquad t \in [t_0,t_r), \quad a(t_r) = a_r,$$

$$b(t) = \sum_{i=1}^{r} \mathrm{I}_{\{t_{i-1}\le t<t_i\}}\left[b_{i-1}\frac{(t_i-t)}{t_i-t_{i-1}} + b_i\frac{(t-t_{i-1})}{t_i-t_{i-1}}\right], \qquad t \in [t_0,t_r), \quad b(t_r) = b_r.$$

Paper is organized as follows. In section 2 we introduce the model with time-varying amplitude of deterministic cycle. In section 3 we investigate the Bayesian inference for such model and we show the closed form for marginal posterior distribution for frequency vector. In the last section we consider real data example concerning monthly industrial production in Poland. Note that in the empirical analyses of the business cycle the choice of industrial production series of monthly frequency seems standard.

2 Model formulation

We consider the following autoregressive model of order p:

$\Psi(L)(Y_t - g(\lambda,t) - \mu) = \varepsilon_t$,(2)

with time-varying mean function $g(\lambda,t)+\mu$, where $\Psi(L) = 1 - \sum_{k=1}^{p}\psi_k L^k$ is a standard polynomial in autoregressive part, $g(\lambda,t)$ is of the form (1) and ε_t is a white noise. Note that equivalently:

$Y_t = \sum_{k=1}^{p}\psi_k Y_{t-k} + \Psi(L)[g(\lambda,t)+\mu] + \varepsilon_t$.

In the next part of this section we show that the above model can be approximated by

$$Y_t = \sum_{k=1}^{p} \psi_k Y_{t-k} + \tilde{g}(\lambda, t) + \tilde{\mu} + \varepsilon_t,$$

where $\tilde{g}(\lambda, t)$ is of the general form (1).

To show this we consider a linear function $s(t)\colon \mathbb{Z} \to \mathbb{R}$ passing through the points (x, z_s) and (y, w_s). In such a case we have

$$s(t) = \frac{t(-(w-z))}{x-y} - \frac{yz - wx}{x-y},$$

$$\begin{aligned}\Psi(L)s(t)\sin(\lambda t) &= \left[1 - \sum_{k=1}^{p} \psi_k L^k\right] s(t)\sin(\lambda t) = s(t)\sin(\lambda t) - \sum_{k=1}^{p} \psi_k s(t-k)\sin(\lambda(t-k)) \\ &= w_s \frac{(x-t)[\sin(\lambda t) - \sum_{k=1}^{p} \psi_k \sin(\lambda(t-k))] - \sum_{k=1}^{p} k\sin(\lambda(t-k))\psi_k}{x-y} \\ &+ z_s \frac{(t-y)[\sin(\lambda t) - \sum_{k=1}^{p} \psi_k \sin(\lambda(t-k))] + \sum_{k=1}^{p} k\sin(\lambda(t-k))\psi_k}{x-y}.\end{aligned}$$

Using elementary trigonometric identities we obtain

$$\sum_{k=1}^{p} \psi_k \sin(\lambda(t-k)) = \sin(\lambda t) f_{ss} + \cos(\lambda t) f_{sc}$$
$$\sum_{k=1}^{p} k\psi_k \sin(\lambda(t-k)) = \sin(\lambda t) g_{ss} + \cos(\lambda t) g_{sc}$$
$$\sum_{k=1}^{p} \psi_k \cos(\lambda(t-k)) = \sin(\lambda t) f_{cs} + \cos(\lambda t) f_{cc}$$
$$\sum_{k=1}^{p} k\psi_k \cos(\lambda(t-k)) = \sin(\lambda t) g_{cs} + \cos(\lambda t) g_{cc},$$

where $f_{ss}, f_{sc}, f_{cc}, f_{cs}, g_{ss}, g_{sc}, g_{cc}, g_{cs}$ depends only on $\lambda, \psi_1, \psi_2, \dots, \psi_p$. In the same way we can decompose $\Psi(L)c(t)\cos(\lambda t)$, where $c(t)\colon \mathbb{Z} \to \mathbb{R}$ is a linear function, passing through the points (x, z_c) and (y, w_c). Using now elementary algebra we get

$$\begin{aligned}&\Psi(L)[s(t)\sin(\lambda t) + c(t)\cos(\lambda t)] = \tfrac{x-t}{x-y}\sin(\lambda t)[1 - f_{ss} - f_{cs}](w_s + w_c) \\ &+ \tfrac{t-y}{x-y}\sin(\lambda t)[1 - f_{ss} - f_{cs}](z_s + z_c) + \tfrac{z_s - w_s}{x-y}[\sin(\lambda t) g_{ss} + \cos(\lambda t) g_{sc}] \\ &+ \tfrac{x-t}{x-y}\cos(\lambda t)[1 - f_{cc} - f_{sc}](w_s + w_c) + \tfrac{t-y}{x-y}\cos(\lambda t)[1 - f_{cc} - f_{sc}](z_s + z_c) \\ &+ \tfrac{z_c - w_c}{x-y}[\sin(\lambda t) g_{cs} + \cos(\lambda t) g_{cc}].\end{aligned} \tag{3}$$

Note that

$$\tfrac{z_s - w_s}{x-y}[\sin(\lambda t) g_{ss} + \cos(\lambda t) g_{sc}] = \left(\tfrac{x-t}{x-y} + \tfrac{t-y}{x-y}\right)\tfrac{z_s - w_s}{x-y}[\sin(\lambda t) g_{ss} + \cos(\lambda t) g_{sc}]$$
$$\tfrac{z_c - w_c}{x-y}[\sin(\lambda t) g_{cs} + \cos(\lambda t) g_{cc}] = \left(\tfrac{x-t}{x-y} + \tfrac{t-y}{x-y}\right)\tfrac{z_c - w_c}{x-y}[\sin(\lambda t) g_{cs} + \cos(\lambda t) g_{cc}].$$

Hence, $\Psi(L)[s(t)\sin(\lambda t) + c(t)\cos(\lambda t)]$ can be equivalently written as

$$\Psi(L)[s(t)\sin(\lambda t)+c(t)\cos(\lambda t)] \quad =\frac{x-t}{x-y}\sin(\lambda t)\widetilde{w}_s+\frac{t-y}{x-y}\sin(\lambda t)\tilde{z}_s \\ +\frac{x-t}{x-y}\cos(\lambda t)\widetilde{w}_c+\frac{t-y}{x-y}\cos(\lambda t)\tilde{z}_s, \tag{4}$$

with bijective transformation $(w_s, w_c, z_s, z_c) \to (\widetilde{w}_s, \widetilde{w}_c, \tilde{z}_s, \tilde{z}_c)$, under constant $\lambda, \psi_1, \psi_2, \dots, \psi_p$. Based on (4) we can approximate the model (2) as

$$Y_t = \sum_{k=1}^{p} \psi_k Y_{t-k} + \tilde{g}(\lambda, t) + \tilde{\mu} + \varepsilon_t,$$

where $\tilde{g}(\lambda, t)$ is of the general form (1). We called the above model approximation since (4) holds on integer line, while we consider in the model a linear spline. The above model can be generalized in natural way to multi-frequency case

$$Y_t = \sum_{k=1}^{p} \psi_k Y_{t-k} + \sum_{k=1}^{m} \tilde{g}_k(\lambda_k, t) + \tilde{\mu} + \varepsilon_t, \tag{5}$$

where $\tilde{g}_k(\lambda_k, t)$, for $k = 1,2, \dots, m$ are of the general form (1). In such a case we use notation $\{(t_{i,k}, a_{i,k}), i = 0,1, \dots, r_k\}$ for $a_k(t)$ and $\{(t_{i,k}, b_{i,k}), i = 0,1,2, \dots, r_k\}$ for $b_k(t)$, where k corresponds to frequency λ_k, $k = 1,2, \dots, m$. To claryfy the model with time-varying amplitude we consider the following ilustrative example, where the dynamics of $g(\lambda, t)$ are ilustrated.

Example 1. We consider $n = 156$, $p = 0$, one frequency $\lambda = 0.15$, $r \in \{2,3,4,6\}$ and with equally spaced $t_0 = 1, t_1, t_2, \dots, t_r = n$. For fixed r we draw each $a_0, a_1, a_2, \dots, a_r$ from uniform distribution on the interval $[2,15]$ and $b_0, b_1, b_2, \dots, b_r$ from uniform distribution on the interval $[-5,0]$.

Table 1. Parameters used in example.

$\boldsymbol{r}$	$\{\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_r}\}$	$\{\boldsymbol{b_0}, \boldsymbol{b_1}, \boldsymbol{b_2}, \dots, \boldsymbol{b_r}\}$
$r = 2$	$\{8.9,2.2,5.8\}$	$\{-0.4, -2., -4.5\}$
$r = 3$	$\{14.7,8.3,3.2,14.\}$	$\{-0.8, -3.9, -2.4, -0.7\}$
$r = 4$	$\{4.7,10.,8.9,2.6,2.9\}$	$\{-4.5,0., -1.4, -2.2, -4.1\}$
$r = 6$	$\{10.2,8.8,8.,4.7,3.4,8.5,6.8\}$	$\{-1., -1.4, -3.4, -2.3, -2.1, -3.6, -4.2\}$

The main finding from presented example (see Fig. 1) is that the cycle based on (1) with time-varying amplitude and with one frequency is much more flexible than deterministic cycle with one frequency and constant amplitude. Hence, the proposed deterministic cycle model with time-varying amplitude may be useful from practical point of view in statistical inference concerning amplitude and length of the cycle.

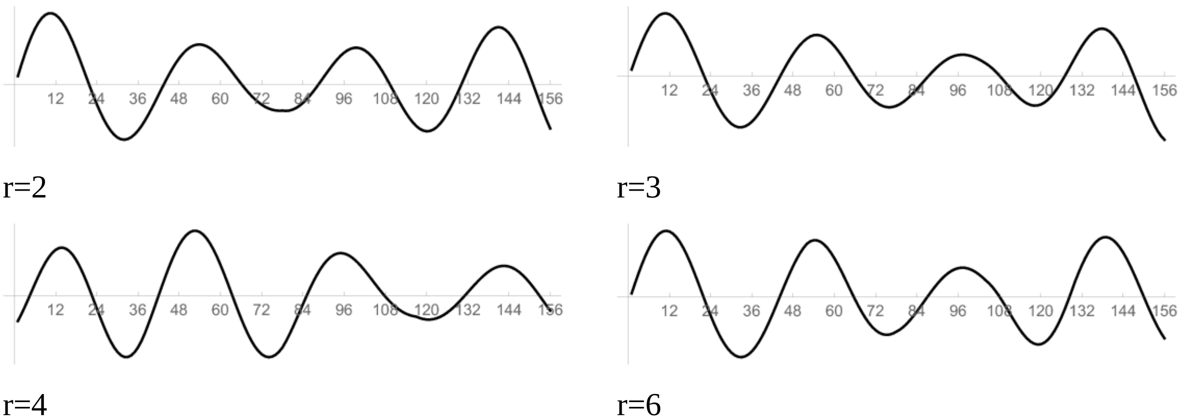


Fig. 1. Paths for $g(\lambda, t)$ for different r and $\{a_0, a_1, a_2, \dots, a_r\}$, $\{b_0, b_1, b_2, \dots, b_r\}$.

3 Bayesian inference for frequency

In this section we obtain closed form of marginal posterior distribution for vector of frequency $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$. Note that model (5) can be equivalently written as:

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon, \tag{6}$$

where $\mathbf{y} = [y_1 \;\; y_2 \;\; \dots \;\; y_n]^{\mathrm{T}}$, the matrix $\mathbf{X}$ depends on $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ and the first coordinates of knots, β is a $\big(1 + p + 2(r_1 + r_2 + \dots + r_{\mathrm{m}} + m)\big)$ x 1 vector of parameters:

$$\beta = \;\; [\mu \;\; \psi_1 \;\; \psi_2 \;\; \dots \;\; \psi_p \;\; a_{0,1} \;\; a_{1,1} \;\; \dots \;\; a_{r_1,1} \;\; b_{0,1} \;\; b_{1,1} \;\; \dots \;\; b_{r_1,1} \;\; \dots$$
$$a_{0,m} \;\; a_{1,m} \;\; \dots \;\; a_{r_m,m} \;\; b_{0,m} \;\; a_{1,m} \;\; \dots \;\; b_{r_m,m}]^{\mathrm{T}}.$$

Moreover we assume that $\varepsilon_t \sim N(0, \tau^{-1})$, for $t = 1, 2, \dots, n$, where $\varepsilon = [\varepsilon_1 \;\; \varepsilon_2 \;\; \dots \;\; \varepsilon_n]^{\mathrm{T}}$. Let us denote $\theta = (\beta, \tau, \lambda_1, \lambda_2, \dots, \lambda_m)$. The likelihood function is of the form:

$$p(\mathbf{y}|\theta) = \frac{1}{\sqrt{(2\pi)^n}} \tau^{\frac{n}{2}} \exp\left\{-\frac{\tau}{2}(\mathbf{y} - \mathbf{X}\beta)'(\mathbf{y} - \mathbf{X}\beta)\right\}.$$

We assume the standard conjugate family of distributions:

$p(\theta) = p(\beta, \tau)p(\varphi) = p(\beta|\tau)p(\tau)p(\varphi)$,

where $\beta|\,\tau \sim N(\mathbf{b}, (\tau\mathbf{B})^{-1})$ and $\tau \sim G\left(\frac{n_0}{2}, \frac{s_0}{2}\right)$, with prior hyperparameters $\mathbf{b}, \mathbf{B}, n_0, s_0$. Under such notation:

$$p(\beta|\tau) = (2\pi)^{-k/2} (\det(\mathbf{B}))^{1/2} \tau^{k/2} \exp\left\{-\frac{\tau}{2}(\beta - \mathbf{b})'\mathbf{B}(\beta - \mathbf{b})\right\},$$

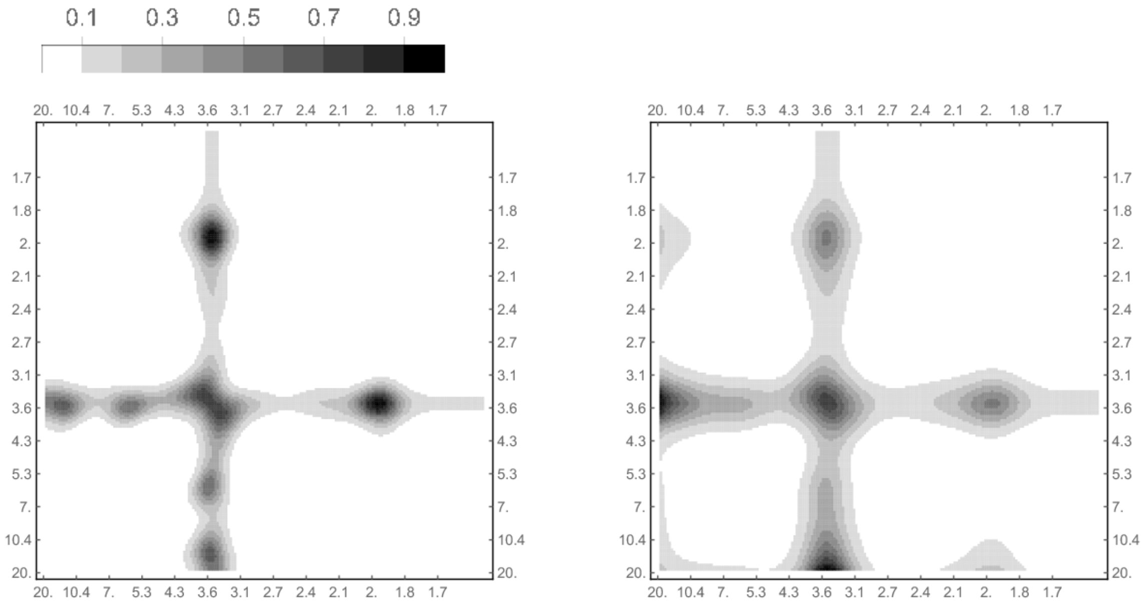
$$p(\tau) = \frac{(s_0/2)^{\frac{n_0}{2}}}{\Gamma\left(\frac{n_0}{2}\right)} \tau^{\frac{n_0}{2} - 1} \exp\left(-\frac{s_0\tau}{2}\right).$$

For the frequency vector $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ we assume uniform distribution on $[\lambda_L, \lambda_U]^m$. Using now the same arguments as in in Lenart and Mazur (2016) we get the marginal posterior distribution for Λ of the form

$$p(\Lambda|\mathbf{y}) \propto (\det(\mathbf{X}'\mathbf{X}+\mathbf{B}))^{-1/2}(\mathbf{y}'[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{B})^{-1}\mathbf{X}']\mathbf{y}+s_0)^{-\frac{n+n_0}{2}}.) \quad (7)$$

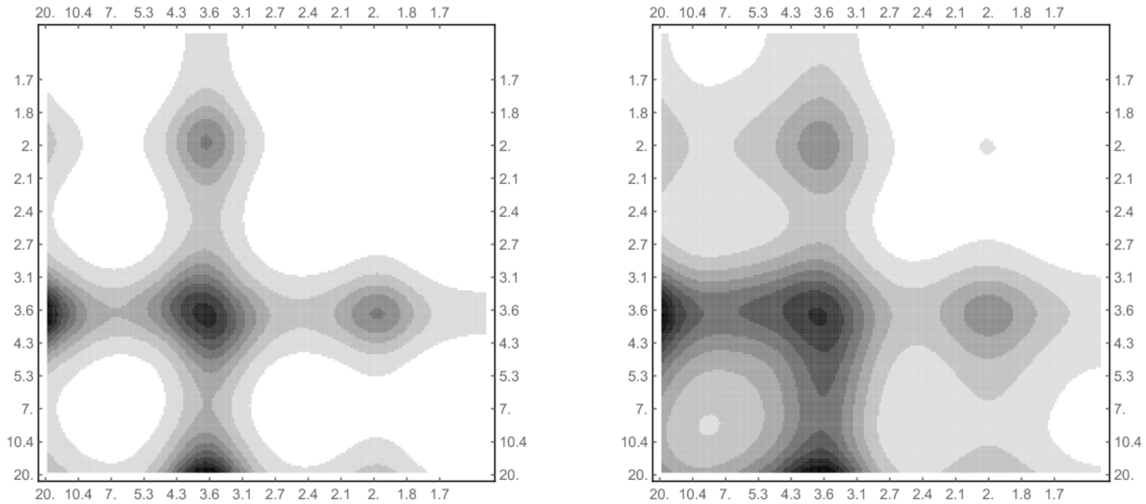
4 Real data example

We consider production in industry in Poland[2] (mining and quarrying; manufacturing; electricity, gas, steam and air conditioning supply – percentage change compared to same period in previous year, calendar adjusted data, not seasonally adjusted data) from Jan. 2001 to Oct. 2017. We restrict attention to the set of frequencies $[\frac{\pi}{120}, \frac{\pi}{9}]$. It means that we consider only the cycles which are not shorter than one and a half year and longer than 20 years. We fix $n = 180$ and we consider last 15 years for the data set in our empirical analysis. Fig. 2 presents the marginal posterior distributions (7) for bivariate case ($\Lambda = (\lambda_1, \lambda_2)$) for constant amplitude, $r = 1$, $r = 2$, $r = 3$ and equally spaced knots. Only the case $p = 10$ and $m = 2$ is presented.



(a) Constant amplitude (b) r=1

[2]Source: Eurostat.



(c) r=2 (d) r=3

Fig. 2. Marginal posterior distribution (kernel) for $\Lambda = (\lambda_1, \lambda_2)$. Horizontal and vertical axis - length of the cycle in years, $p = 10$. Shades of grey corresponds to kernel value.

(a) Constant amplitude (b) r=1

(c) r=2 (d) r=3

Fig. 3. Marginal posterior distribution (kernel) for one frequency λ_1 in bivariate case $\Lambda = (\lambda_1, \lambda_2)$. Horizontal axis - length of the cycle in years, $p = 10$.

Marginal distribution (kernel) under constant amplitude (see Fig. 2 (a)) show two predominant length of the deterministic cycle. Cycle with length approx. 3.5 year and 2 years (see also Fig. 3). The mass is concentrated relativly close these two frequencies compared to time-varying amplitude (see Fig. 2 (b)-(d)). In the case of time-varying amplitude the mass is still concentrated near these frequencies, but with greater dispersion, especially for $r = 3$. Hence, the bigger r the weaker multimodality of posterior distribution. The main findings is that for each considered r the results clearly support the length of the business cycle approx. 3.5 years (see Fig. 3). The similar conclusions can be found in Lenart et al. (2016), where the same length of the cycle was detected using industrial production in Poland to Dec. 2014.

Conclusions

The closed form of marginal posterior distribution for frequencies in case of time-varying amplitude was shown. This gives the opportunity to expand the statistical inference proposed in Lenart and Mazur (2016). The real data example shows that in the case of last 15 years of industrial production in Poland (covering the period 2002-2017) the predominant length of the cycle is approx. 3.5 year. This conclusion is supported by both constant amplitude case and time-varying case. An open problem is the construction of the forecast assuming a time-varying amplitude of the deterministic cycle.

Acknowledgements

This research was financed from the funds granted to the Faculty of Finance and Law at Cracow University of Economics, within the framework of the subsidy for the maintenance of research potential.

References

Azevedo, J. V., Koopman, S. J. & Rua, A. (2006). Tracking the business cycle of the euro area: a multivariate model-based band-pass filter. *Journal of Business & Economic Statistics*, *24*(3), 278–290.

Harvey, A. C. (2004). Tests for cycles. In: Harvey, A.C., Koopman, S.J. & Shephard, N. (eds), *State space and unobserved component models, CUP*, pages 102–119.

Harvey, A. C. & Jaeger, A. (1993). Detrending, stylized facts and the business cycle. *Journal of Applied Econometrics*, *8*, 231–247.

Harvey, A. C. & Trimbur, T. M. (2003). General model-based filters for extracting cycles and trends in economic time series. *Review of Economics and Statistics, 85*(2), 244–255.

Harvey, A. C., Trimbur, T. M. & Van Dijk, H. K. (2007). Trends and cycles in economic time series: A bayesian approach. *Journal of Econometrics*, *140*, 618–649.

Koopman, S. J. & Shephard, N. (2015). *Unobserved Components and Time Series Econometrics*. Oxford university Press, Oxford.

Lenart, Ł. & Mazur, B. (2016), On Bayesian estimation of almost periodic in mean autoregressive models. *Przegląd Statystyczny*, *63*(3), 255–271.

Lenart, Ł., Mazur, B. & Pipień, M. (2016). Statistical analysis of business cycle fluctuations in Poland before and after the crisis. *Equilibrium. Quarterly Journal of Economics and Economic Policy*, *11*(4), 769–783.

Lenart, Ł. & Pipień, M. (2013). Almost Periodically Correlated Time Series in Business Fluctuations Analysis. *Acta Physica Polonica A*, *123*, 567–583.

Lenart, Ł. & Pipień, M. (2015). Własności empiryczne cyklu finansowego - analiza porównawcza Czech, Polski, Węgier, Wielkiej Brytanii i USA. *Folia Oeconomica Cracoviensia*, *56*, 81–112.

Lenart, Ł. & Pipień, M. (2017a). Non-Parametric Test for the Existence of the Common Deterministic Cycle: The Case of the Selected European Countries. *Central European Journal of Economic Modelling and Econometrics*, *9*(3), 201–241.

Lenart, Ł. & Pipień, M. (2017b). The Dynamics of the Credit Cycle in Selected Asian Countries. *Folia Oeconomica Cracoviensia*, *58*, 127-133.

Mazur, B. (2016). Growth cycle analysis: the case of Polish monthly macroeconomic indicators. *Folia Oeconomica Cracoviensia*, *57*, 37–54.

Mazur, B. (2017a). Probabilistic predictive analysis of business cycle fluctuations in Polish economy. *Equilibrium. Quarterly Journal of Economics and Economic Policy*, *12*(3),435–452.

Mazur, B. (2017b). Probabilistic Prediction Using Disaggregate Data: The Case of Value Added in Poland. *Folia Oeconomica Cracoviensia*, *58*, 85–103.

Pelagatti M. M. (2016). *Time Series Modelling with Unobserved Components*. Taylor & Francis Group, Boca Raton.

Trimbur T. M. (2006). Properties of higher order stochastic cycles. *Journal of Time Series Analysis*, *27*, 1–17.