A family of non-standard bivariate distributions with applications to unconditional modelling in empirical finance

Błażej Mazur[1], Mateusz Pipień[2]

Abstract

We develop a class of parametric bivariate distributions that are capable of accounting for non-standard empirical properties that are evident in some financial time series. We aim at creating a parametric framework that allows for serious divergences from the multivariate Gaussian case both in terms of properties of marginal distributions and in terms of the dependence pattern. We are particularly interested in obtaining a multivariate construct that allows for considerable degree of heterogeneity in marginal properties of its components (like tail thickness and asymmetry). Moreover, we consider non-standard dependence patterns that go beyond a linear correlation-type relationship while maintaining simplicity, obtained by introducing rotations. We make use of marginal distributions that belong to generalized asymmetric *t* class analysed by Harvey and Lange (2017), allowing not only for skewness but also for asymmetric tail thickness. We illustrate flexibility of the resulting bivariate distribution and investigate its empirical performance examining unconditional properties of bivariate daily financial series representing dynamics of stock price indices and the related FUTURES contracts.

1   Introduction

The empirical distributions of economic variables might display serious divergences from the multivariate normal case; see Genton (2004). For example pervasive heavy tails and volatility amplify studies that aim at searching for an appropriate families of multivariate probability distributions that would lead to successful modelling of empirical features in case of related financial returns. In particular since 2000's many authors tried to go beyond conditional normality assumed commonly in case of Multivariate GARCH (M-GARCH) models. For example conditionally elliptical distribution in DCC model was presented by Pelagatti and Rondena (2004). Some other non-Gaussian conditional distributions we analysed by Bauwens and Laurent (2005), Sahu et al. (2001), Pipień (2012) and others. However, commonly applied econometric strategy using the Maximum Likelihood estimation procedure might

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[1]Corresponding author: Department of Econometrics and Operations Research, Cracow University of Economics, Rakowicka 27, Cracow, Poland; e-mail: eomazur@cyf-kr.edu.pl.
[2]Department of Econometrics and Operations Research, Cracow University of Economics, Rakowicka 27, Cracow, Poland; e-mail: eepipien@cyf-kr.edu.pl.

result with a considerable small-sample bias; see Iglesias and Phillips (2006).There is no doubt that proper modelling of empirical features observed for the case of related financial time series require construction of a flexible class of distributions; moreover, the development of alternative methods of statistical inference is necessary. In this paper we address these two issues. We propose a novel class of parametric bivariate distributions to model empirical properties that are evident in some financial time series. We depart from the multivariate Gaussian case both in terms of properties of the marginal distributions and in terms of the co-dependence pattern. In order to achieve flexibility we make use of marginal distributions that belong to generalized asymmetric Student-*t* class analysed by Harvey and Lange (2017), allowing not only for skewness but also for asymmetric tail thickness. We also develop methods of formal Bayesian inference and present posterior analysis within constructed class of sampling models. We also consider the issue of Bayesian (density) prediction.In the paper we illustrate flexibility of the resulting bivariate distribution and investigate its empirical performance examining unconditional properties of bivariate daily financial series representing dynamics of stock price indices and the related FUTURES contracts.

2   A family of non-standard bivariate distributions

For *m*-variate random variable $\mathbf{z} = (z_1, \dots, z_m)$ with independent coordinates, i.e. with $p(\mathbf{z}) = \prod_{i=1}^{m} p_i(z_i)$, one may consider a distribution resulting from a linear transformation:

$$\boldsymbol{\varepsilon} = A \cdot \mathbf{z} + \mathbf{b}.$$

For a nonsingulartransformation matrix $A_{[mxm]}$the distribution of $\boldsymbol{\varepsilon}$ is described by a well-defined density of the following form:

$$p(\boldsymbol{\varepsilon}) = \frac{1}{|\det(A)|} \prod_{i=1}^{m} p_i(A^{-1\,(i)}(\boldsymbol{\varepsilon} - \mathbf{b})),$$

where $A^{-1\,(i)}$ denotes *i*-th row of matrix A^{-1}.In what followswe make use of this result in a bivariate setting, going far beyond the standard scheme where *A* is defined as a root of symmetric and positive definite matrix generating the covariance structure.

To define the univariate distributions $p_i$we apply the generalised Student *t*-distribution proposed recently by Harvey and Lange (2017). It generalizes previous results by Zhu and Zinde-Walsh (2009), Zhu and Galbraith (2010)as well asFernández and Steel(1998) and Theodossiou (1998) among others, see the references and discussion by Harvey and Lange (2017).

However, the form used here is re-scaled to ensure unit variance. The resulting probability density function (with mode at 0 and nonzero mean in general) is:

$$f_{GAST}(z; h, \alpha, v_L, v_R, \eta_L, \eta_R) = \frac{K_{LR}\sqrt{f}}{\sqrt{h}} \begin{cases} \left[1 + \frac{1}{\eta_L}\left(\frac{-z\sqrt{f}}{2\alpha\sqrt{h}}\right)^{v_L}\right]^{-\frac{1+\eta_L}{v_L}}, & z \leq 0 \\ \left[1 + \frac{1}{\eta_R}\left(\frac{z\sqrt{f}}{2(1-\alpha)\sqrt{h}}\right)^{v_R}\right]^{-\frac{1+\eta_R}{v_R}}, & z > 0 \end{cases}$$

where GAST stands for 'generalized asymmetric skew t' and h denotes variance. The distribution is a two-piece version of a generalized t distribution; parameter $0 < \alpha < 1$ introduces skewness, with $\alpha = 0.5$ denoting the absence of skewing (symmetry requires also$\eta_L = \eta_R$ and$v_L = \eta_R$);v's control shape around the mode (being more flat or spiked, in a GED-like manner, with $v = 2$ leading to t-type shape), while η's affect tail thickness (we require that $\eta_L, \eta_R > 2$ to ensure that variance is finite). However, the influence of η's and v's on tail behaviour is not separated clearly. Setting $v_L = v_R = 2$ and $\eta_L = \eta_R$ leads to skew-t case, with skew-normal and normal distributions being the limiting ones with η_L and$\eta_R \to \infty$. Hence, the asymmetric and flexible distribution encompasses a number of well-known distributions, including the GED (η_L and $\eta_R \to \infty$). We assume $\eta_L, \eta_R > 2$ and $v_L, v_R > 1$.K_{LR} and f denote rather complicated functions of shape parameters: K_{LR} and $\alpha^*$is given by Harvey and Lange (2017), note that P{Z < 0} = α^*,and$f = d - c^2$ with:

$$c = -\alpha^*2\alpha \frac{\eta_L^{\frac{1}{v_L}}\Gamma\left(\frac{2}{v_L}\right)\Gamma\left(\frac{\eta_L-1}{v_L}\right)}{\Gamma\left(\frac{\eta_L}{v_L}\right)\Gamma\left(\frac{1}{v_L}\right)} + (1-\alpha^*)2(1-\alpha)\frac{\eta_R^{\frac{1}{v_R}}\Gamma\left(\frac{2}{v_R}\right)\Gamma\left(\frac{\eta_R-1}{v_R}\right)}{\Gamma\left(\frac{\eta_R}{v_R}\right)\Gamma\left(\frac{1}{v_R}\right)},$$

$$d = -\alpha^*4\alpha^2 \frac{\eta_L^{\frac{2}{v_L}}\Gamma\left(\frac{3}{v_L}\right)\Gamma\left(\frac{\eta_L-2}{v_L}\right)}{\Gamma\left(\frac{\eta_L}{v_L}\right)\Gamma\left(\frac{1}{v_L}\right)} + (1-\alpha^*)4(1-\alpha)^2\frac{\eta_R^{\frac{2}{v_R}}\Gamma\left(\frac{3}{v_R}\right)\Gamma\left(\frac{\eta_R-2}{v_R}\right)}{\Gamma\left(\frac{\eta_R}{v_R}\right)\Gamma\left(\frac{1}{v_R}\right)}.$$

Consider a product measure-type bivariate generalised Student t distribution of the form:

$$p(\mathbf{z}) = p(z_1, z_2) = p_{Z_1}(z_1)p_{Z_2}(z_2),$$

where$p_{Z_i}(z_i)$ is the generalised Student t distribution of Harvey and Lange (2017) with individual shape parameters, transformed into the above GAST form (parametrized in terms of variance).The only restriction is that each coordinate in $\mathbf{z}$ has the same variance h_1, i.e. $V(\mathbf{z}) = h_1\mathbf{I}$. The existence of variance could in principle be relaxed (in a scale-driven model) in order to allow for e.g. Cauchy-type tails. Now assume that the variable $\mathbf{z}$is subject to a linear transformation, but the transformationmatrix is orthogonal; i.e.: $\mathbf{v} = \mathbf{R}_{(\varphi)}\mathbf{z}$, where:

$$\mathbf{R}_{(\varphi)} = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix}.$$

The matrix $\mathbf{R}_{(\varphi)}$imposes clockwise rotationby angle φ;$\mathbf{R}_{(\varphi)}^{-1} = \mathbf{R}_{(\varphi)}'$ with$\det\left(\mathbf{R}_{(\varphi)}\right) = 1$; the transformation might affect the density type but leaves the covariance structure

intact.Thus v_i's are uncorrelated with the same variances, but their marginal distribution might change (relative to those of z_i's). The density of the distribution of $\mathbf{v}$ is given by the formula:

$$p(\mathbf{v}) = \det\!\left(\mathbf{R}_{(\varphi)}\right) p_{Z_1}\!\left(\mathbf{R}_{(\varphi)}^{(1)'}\mathbf{v}\right) p_{Z_2}\!\left(\mathbf{R}_{(\varphi)}^{(2)'}\mathbf{v}\right) = p_{Z_1}\!\left(\mathbf{R}_{(\varphi)}^{(1)'}\mathbf{v}\right) p_{Z_2}\!\left(\mathbf{R}_{(\varphi)}^{(2)'}\mathbf{v}\right).$$

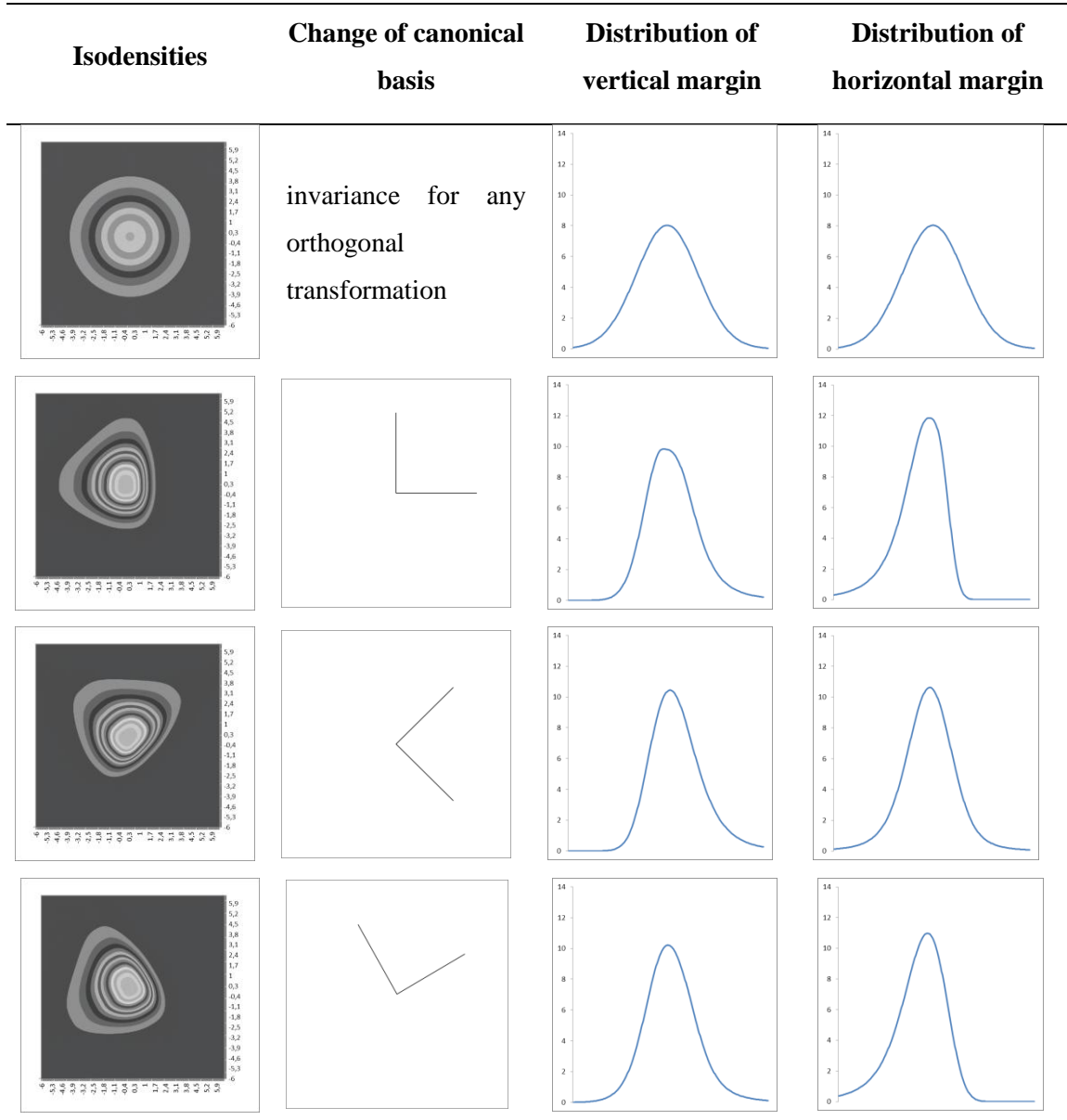| Isodensities	Change of canonical basis	Distribution of vertical margin	Distribution of horizontal margin
	invariance for any orthogonal transformation		

Fig.1. Plots of isodensities, transformation of the canonical basis and marginal distributions of coordinates in Gaussian case (first row) as well as in case of $\mathbf{v} = \mathbf{R}_{(\varphi)}\mathbf{z}$, for $\varphi = 0$ (second row), $\varphi = \frac{\pi}{4}$ (third row), $\varphi = -\frac{\pi}{6}$ (fourth row).

Fig. 1 shows how the shape of the isodensities of $\mathbf{v}$ varies with respect to different values of the shape and the asymmetry parameters. In each case we analyse distributions with variances for margins equal to 4. In the first row we plotted the reference case as the bivariate

Gaussian distribution, being a limiting case here. Possible directional asymmetry and different tail behaviour is presented by the isodensities in the second row. The effect of rotation by a different angle is shown in the third and fourth row. The most interesting property of the analysed distributions is that the dependence pattern assumes zero correlations.
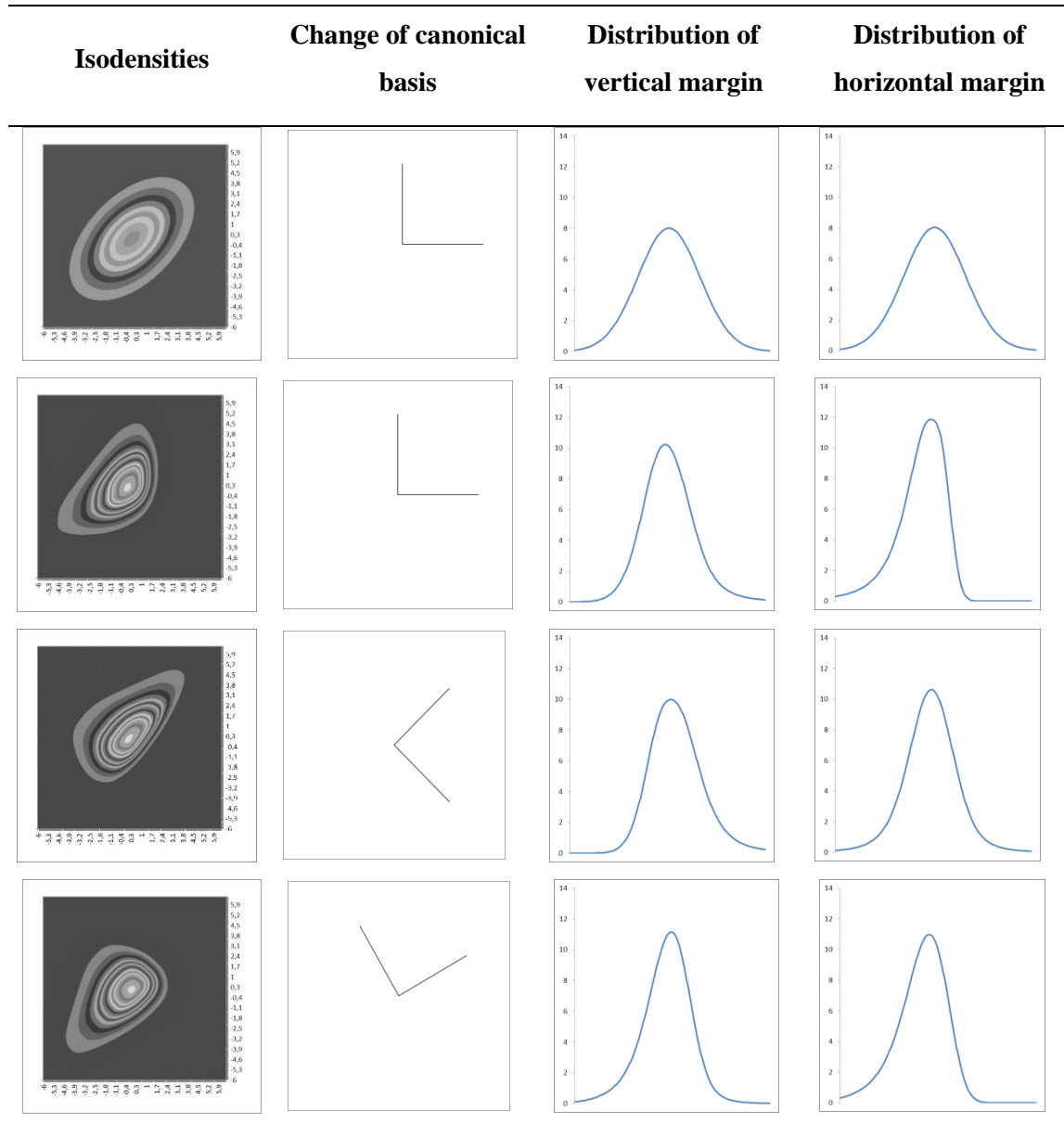
Isodensities	**Change of canonical basis**	**Distribution of vertical margin**	**Distribution of horizontal margin**

Fig.2. Plots of isodensities, transformation of the canonical basis and marginal distributions of coordinates in Gaussian case (first row) as well as in case of $\mathbf{y} = \mathbf{Cv}$ for $\varphi = 0$ (second row), $\varphi = \frac{\pi}{4}$ (third row), $\varphi = -\frac{\pi}{6}$ (fourth row). In each case we keep the same variances for margins equal to 4 and correlation $\rho = 0.5$.

The (rotated) vector $\mathbf{v}$ is subject to a further linear transformation that imposes location (mode)$\mathbf{m}$and covariance structure (i.e. correlation and differences in variances) upon $\mathbf{v}$:

$$\mathbf{y} = \mathbf{Cv} + \mathbf{m}.$$

The matrix $\mathbf{C}$ can be parametrized using different concepts of matrix roots, though here we assume that it has the following form:

$$\mathbf{C} = \begin{bmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & \frac{\sqrt{h_2}}{\sqrt{h_1}} \end{bmatrix}.$$

Then h_i are variances of coordinates and $\rho \in (-1,1)$ represents the correlation coefficient between y_1 and y_2. The density of the distribution of y is given by the following formula:

$$p(\mathbf{y}) = \det(\mathbf{C}^{-1})p_{Z_1}\left(\mathbf{R}_{(\varphi)}^{(1)'}\mathbf{C}^{-1}(\mathbf{y}-\mathbf{m})\right)p_{Z_2}\left(\mathbf{R}_{(\varphi)}^{(2)'}\mathbf{C}^{-1}(\mathbf{y}-\mathbf{m})\right).$$

Fig. 2 depicts isodensities of some exemplary cases of distribution of $\mathbf{y} = \mathbf{Cv}$. We analyse correlated versions of distributions presented on Fig. 1. In each case we assumed correlation coefficient $\rho = 0.5$. Bivariate distributions presented on Fig. 2 show remarkable degree of flexibility in modelling structure of observables, though it does not exceed the case of a linear transformation of the product measure initially defined for a vector $p(\mathbf{z})$. Its flexibility results from the fact that all the shape parameters could be made dimension-specific (in the space of z's). Crucially, the original formulation of Harvey and Lange allows for a complicated asymmetry pattern which is here generalized to a higher dimension. The distribution is unimodal by construction (which is not necessarily true about some other flexible constructs like mixtures), however its mean is a complicated function of all the model parameters. The rotation angle $\varphi$is identified if $p(\mathbf{z})$ defines a distribution class that is not closed under rotations, which holds almost everywhere in the parameter space considered here. However, e.g. for a (limiting) special case of bivariate Gaussian distribution, φ would be locally unidentified.

3   Empirical illustration

We analysed daily logarithmic returns of the S&P500 SPOT and FUTURES together with volumes traded, covering the period from 28.08.2001 till 12.12.2017; 4099 observations. We considered four bivariate datasets, namely the daily returns of the SPOT index with daily returns of the FUTURES volume traded (dataset A), the daily returns of the SPOT index with daily returns of the SPOT volume traded (dataset B), the daily returns of the SPOT index with daily returns of the FUTURES index (dataset C) and the daily returns of the SPOT volume

traded with the daily returns of the FUTURES volume traded (dataset D). The empirical; distribution of analysed bivariate series are presented on Fig. 3.

We applied the class of bivariate distributions, presented in the previous part to model the unconditional distribution of analysed series. In order to perform this task we constructed Bayesian models for each analysed series.The estimation was carried over using the Metropolis-Hastings Random Walk sampler; we assume prior independence across all the model parameters. The priors are informative though tailored to convey relatively weak information, for example for φ, ρ and α we assume uniform priors. A posteriori we find limited skewness (with $\alpha = 0.5$ being rather likely) but strong shape asymmetry (e.g. clear evidence against $v_L = v_R$ or $\eta_L = \eta_R$ in some cases) which justifies the empirical relevance of shape-asymmetric distributions of Zhu and Zinde-Walsh (2009), Zhu and Galbraith (2010)as well as Harvey and Lange (2017). We find support for $v_L < 1.5$ (e.g. using SPOT index returns), $2 < \eta_L < 3$ e.g. using volume growth rates of FUTURES; full estimation results are available from the authors upon request.
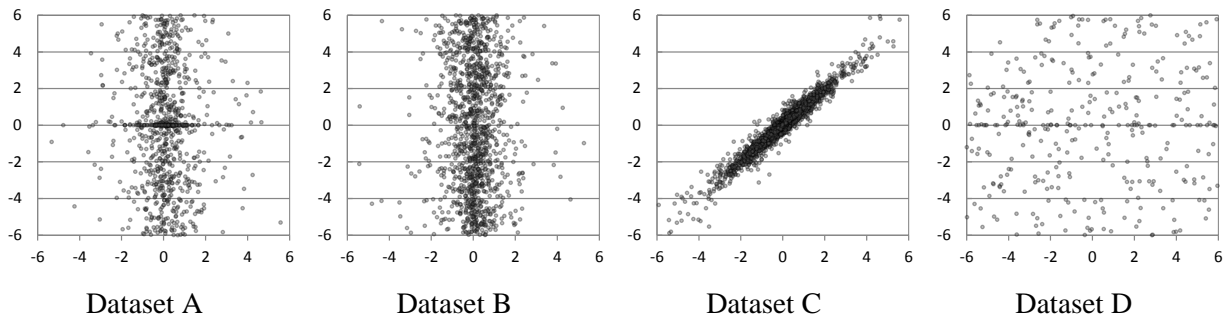

Fig.3. Analysed bivariate time series (the axes are adjusted to match Fig. 4).

The estimated unconditional distributions are presented on Fig. 4. For each dataset we plotted isodensities of the distribution corresponding to$p(\mathbf{y})$with posterior means of parameters used as plug-in estimates. We report the empirical importance of the rotation effect (relying on the posterior mean of parameter φ). The shapes of resulting marginal univariate distributions are also presented.In case of datasets A and B we see a little data support in favour of dependence. Also the rotation effect seems negligible. The posterior mean of parameter φ is equal to 0.068 in case of dataset A and to 0.050 in case of dataset B. Also both datasets support small negative correlation, indicating no substantial linear dependence between the variables. The posterior mean of the correlation ρ= - 0.089 in case of dataset A and $\rho = -0.057$ for dataset B.
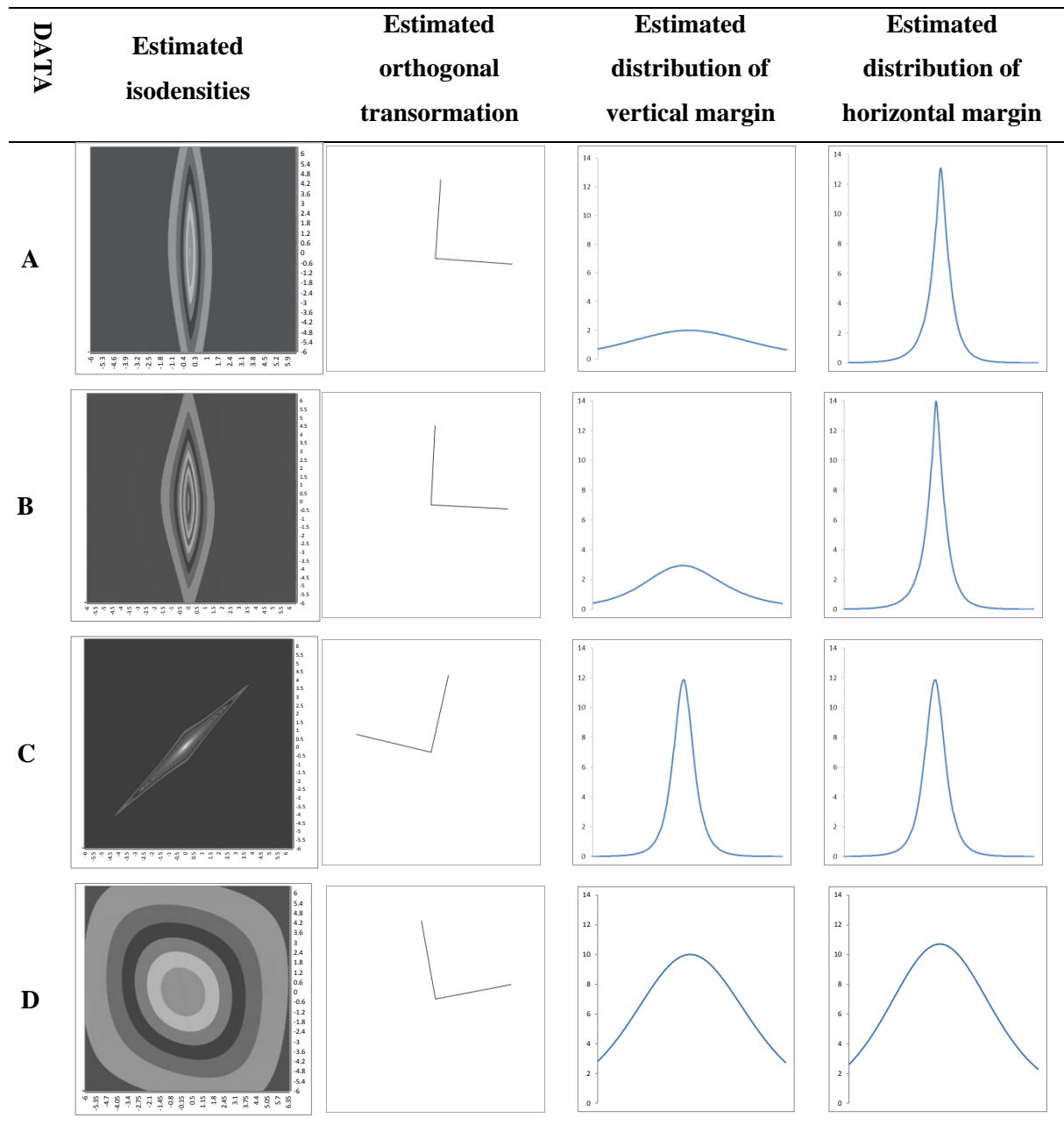
DATA	Estimated isodensities	Estimated orthogonal transormation	Estimated distribution of vertical margin	Estimated distribution of horizontal margin
A				
B				
C				
D				

Fig. 4. Estimation results: plots of isodensities and margins for the parameter values corresponding to posterior means.

The strong linear dependence as well as empirical importance of the rotation effect was obtained for the case of dataset C. Estimated posterior mean of parameter $\varphi = -1.345$ indicates strong counter clockwise rotation of coordinates, by more than 75 degrees. The posterior mean of correlation parameter $\rho = 0.969$ is rather high in this case. A moderate effect of dependence was obtained in case of dataset D. We report some evidence in favour of the rotation effect, as the posterior mean of $\varphi = -0.185$. It results with counter clockwise rotation of coordinates by about 10 degrees. The dataset D can be also described by small positive

correlation, since the posterior mean of ρ=0.236. The posterior-predictive distribution (that takes into account the estimation uncertainty) is depicted in Fig. 5 (for the dataset B).
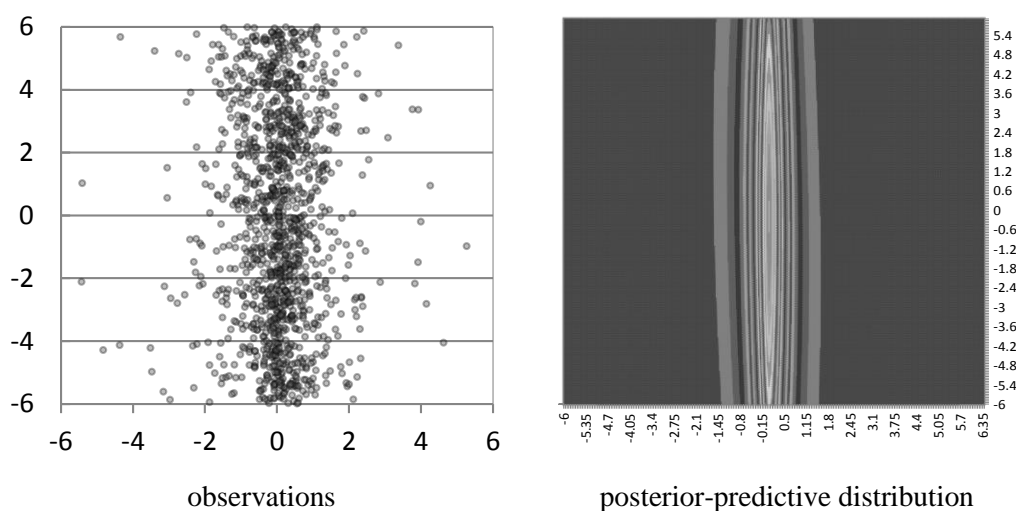

Fig. 5. Dataset B: the data versus the posterior-predictive density.

Conclusions

We develop a class of flexible multivariate distributions that differs from the usual ones in two particular aspects. Firstly, we allow for high degree of stochastic heterogeneity across variables (addressing asymmetry and tail thickness issues), allowing for 5 shape parameters per dimension. Secondly, we introduce the dependence not only via covariances but also via rotations (which is possible due to generality of the distribution). This approach differs from the alternative ones e.g. using the copula functions: here the form of marginal distribution is not directly controlled. However, the dependence structure is imposed by a well-defined transformation that goes beyond considering covariances only while it remains tractable also in higher dimensions. Hence we provide a practical generalization of a product measure which allows for high degree of heterogeneity, more complicated dependence while avoiding potential problems that arise within high-dimensional modelling using copula functions. The number of shape parameters increases linearly with dimension, but one could of course consider less-heavily parametrized special cases obtained by linear constraints. Therefore the model provides ageneral framework allowing for the search for empirically relevant (restricted) special cases. Importantly, the construct considered here could be used to define conditional distribution in a dynamic model, which will be subject to further research.

Acknowledgements

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